Review of Working with Negations

These notes include

- A review of basic facts about negations
- Some examples, including one very complex example
- Some practice problems, and
- The solutions to the practice problems (a link to the solution is provided with each problem).

1. Basic Facts

The six most important rules for working with negations of statements:

(0) $\sim\sim p$ is equivalent to p
(1) $\sim(p \wedge q)$ is equivalent to $\sim p \vee \sim q$
(2) $\sim(p \vee q)$ is equivalent to $\sim p \wedge \sim q$
(3) $\sim(p \rightarrow q)$ is equivalent to $p \wedge \sim q$
(4) $\sim(\forall x)P(x)$ is equivalent to $(\exists x)\sim P(x)$
(5) $\sim(\exists x)P(x)$ is equivalent to $(\forall x)\sim P(x)$

The four basic rules about negations of inequalities:

(6) $\sim(a < b)$ is equivalent to $a \geq b$
(7) $\sim(a \leq b)$ is equivalent to $a > b$
(8) $\sim(a > b)$ is equivalent to $a \leq b$
(9) $\sim(a \geq b)$ is equivalent to $a < b$

Finally, just for completeness, two rules about equality:

(10) $\sim(a = b)$ is equivalent to $a \neq b$
(11) $\sim(a \neq b)$ is equivalent to $a = b$

2. Some Examples that involve applying Two or More Rules At Once

E1. What is the negation of 5 < x < 7?

Solution: $5 < x < 7$ is a shorthand for $(5 < x) \wedge (x < 7)$.

First apply rule (1)

$\sim((5 < x) \wedge (x < 7))$ is equivalent to $\sim (5 < x) \vee \sim (x < 7)$

Next, apply (6) twice:

$\sim (5 < x) \vee \sim (x < 7)$ is equivalent to $(5 \geq x) \vee (x \geq 7)$

This can be more conveniently written as

$(x \leq 5) \vee (x \geq 7)$

E2. What is the negation of "There is a real number whose square is less than –1"?

First, lets write this symbolically:

$(\exists\, x \in \mathbf{R})\ (x^2 < -1)$

Its negation is

$\sim (\exists\, x \in \mathbf{R})\ (x^2 < -1)$

Using rule (5), this is equivalent to

$(\forall\, x \in \mathbf{R}) \sim (x^2 < -1)$

Using rule (6), we can write this more conveniently as

$(\forall\, x \in \mathbf{R})\ (x^2 \geq -1)$

E3. A very complex statement – and how to deal with complexity.

In more advanced mathematics, one encounters the concept of a "continuous function". Intuitively, a function f: $\mathbf{R} \rightarrow \mathbf{R}$ is continuous if its graph is a single (possible curved) line, with no jumps. $f(x) = x^2$ is continuous. However, the function g defined by

$g(\mathrm{x}) = 0$ if $x \leq 0$

$g(x) = 1$ if $x > 0$

is not continuous – its graph jumps from 0 to 1 as x crosses 0. (Try sketching the graph of this function!)

Although this idea may seem simple, it is not easy to define precisely what we mean by continuity. I'll write down the precise definition, then use it to show that the function g just defined is not continuous.

"g is continuous" means

$$(\forall x)\ (\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall y)\ (\ |y - x| < \delta \rightarrow |g(y) - g(x)| < \varepsilon)$$

(Don't try to get an intuitive feel for what is being said here yet.)

I've taken the liberty to use a couple of Greek letters, in addition to Roman letters, because it is customary to do so when discussing this topic.

The definition has four quantifiers! The only practical way of dealing with a statement this complex is to apply our negation rules, one at a time, very systematically, taking no shortcuts. Since we want to prove that g is *not* continuous, we need to write down "g is not continuous" in a way that we can work with it.

"g is not continuous" means:

$$\sim (\forall x)\ (\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall y)\ (\ |y - x| < \delta \rightarrow |g(y) - g(x)| < \varepsilon)$$

Apply rule (4) to move the negation sign inside the first universal quantifier. We get

$$(\exists x) \sim (\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall y)\ (\ |y - x| < \delta \rightarrow |g(y) - g(x)| < \varepsilon)$$

Apply rule (4) again:

$$(\exists x)\ (\exists \varepsilon > 0) \sim (\exists \delta > 0)\ (\forall y)\ (\ |y - x| < \delta \rightarrow |g(y) - g(x)| < \varepsilon)$$

Apply rule (5) to move the negation sign inside the existential quantifier:

$$(\exists x)\ (\exists \varepsilon > 0)\ (\forall \delta > 0) \sim (\forall y)\ (\ |y - x| < \delta \rightarrow |g(y) - g(x)| < \varepsilon)$$

Apply rule (4) for a third time:

$$(\exists x)\ (\exists \varepsilon > 0)\ (\forall \delta > 0)\ (\exists y) \sim (\ |y - x| < \delta \rightarrow |g(y) - g(x)| < \varepsilon)$$

Now apply rule (3), which tells us how to form the negation of a conditional statement:

$$(\exists x)\ (\exists \varepsilon > 0)\ (\forall \delta > 0)\ (\exists y)\ (\ |y - x| < \delta \wedge \sim |g(y) - g(x)| < \varepsilon)$$

Finally, apply rule (6):

$$(\exists x)\ (\exists \varepsilon > 0)\ (\forall \delta > 0)\ (\exists y)\ (\ |y - x| < \delta \wedge\ |g(y) - g(x)| \geq \varepsilon)$$

This reads as follows

"There are a number x, and a positive number ε, such that for every positive number δ, there is a number y such that $|y - x| < \delta$ and $|g(y) - g(x)| \geq \varepsilon$."

To demonstrate the truth of this, we have to exhibit x and ε with the required properties. I propose $x = 0$ and $\varepsilon = ½$. Plugging in these values, what we need to show is:

$$(\forall\delta > 0)\ (\exists y)\ (\ |y - 0| < \delta \wedge\ |g(y) - g(0)| \geq ½\ )$$

Since $g(0) = 0$, this can be written more simply as

$$(\forall\delta > 0)\ (\exists y)\ (\ |y| < \delta \wedge\ |g(y)| \geq ½\ )$$

This has only two quantifiers and is manageable! It can be read,

"For every positive number δ, there is a number y such that
$|y| < \delta$ and $|g(y)| \geq ½$

It is easy to see that this is true. Given a positive number δ, choose $y = \delta/2$. Check out both sides of the "and":

$|y| < \delta$ because $\delta/2 < \delta$.
$|g(y)| \geq ½$ because $g(y) = 1$ (refer to the definition of g!)

This completes the proof that g is not a continuous function.

Lesson learned:

Complexity can be managed by taking things one step at a time.

3. Some practice in forming negations, and translating between verbal and symbolic forms

For each of the following statements,

(a) translate it into a symbolic form,
(b) form the negation of the statement, and transform it by applying as many of our rules as possible, and
(c) translate the negation back to a predominantly verbal form.

Incidentally, for the purpose of practicing dealing with negation rules, we don't need to be concerned about whether the statements below are true or false. Some are true, some aren't.

P1. For all real numbers x and y, $x < y$.

Solution

P2. There are two real numbers that are not equal to one another.

Solution

P3. Every real number has a square root.

Solution

P4. The square of any irrational number is irrational. [Note the negation lurking in the *ir* part of *ir*rational!]

Solution

P5. A quadratic equation $ax^2 + bx + c = 0$ always has at least one solution.

Solution

P6. There is a number z such that if you add z to any number y, you get y.

Solution

4. Solutions to the six practice exercises.

P1. For all real numbers x and y, $x < y$.

"For all" means universal quantifiers, and there are two of them:

$$(\forall x \in \mathbf{R})(\forall y \in \mathbf{R})(x < y)$$

Its negation is

$$\sim (\forall x \in \mathbf{R})(\forall y \in \mathbf{R})(x < y)$$

Apply rule (4):

$$(\exists x \in \mathbf{R}) \sim (\forall y \in \mathbf{R})(x < y)$$

Apply (4) again:

$$(\exists x \in \mathbf{R}) (\exists y \in \mathbf{R}) \sim (x < y)$$

Apply rule (6):

$$(\exists x \in \mathbf{R}) (\exists y \in \mathbf{R}) (x \geq y)$$

This can be read, "There exit real numbers x and y such that x is greater than or equal to y."

In the remainder of the solutions, I'll omit citing rules by number.

Next Problem

P2. There are two real numbers that are not equal to one another.

$$(\exists x \in \mathbf{R})\ (\exists y \in \mathbf{R})\ \ (x \neq y)$$

Its negation is

$$\sim (\exists x \in \mathbf{R})\ (\exists y \in \mathbf{R})\ \ (x \neq y)$$

which is equivalent to

$$(\forall x \in \mathbf{R}) \sim (\exists y \in \mathbf{R})\ \ (x \neq y)$$

which is equivalent to

$$(\forall x \in \mathbf{R})\ (\forall y \in \mathbf{R}) \sim (x \neq y)$$

and finally

$$(\forall x \in \mathbf{R})\ (\forall y \in \mathbf{R})\ \ (x = y)$$

which says, "Every real number is equal to every other real number."

Next Problem

P3. Every real number has a square root.

$$(\forall x \in \mathbf{R})\ (\exists y \in \mathbf{R})\ \ (y^2 = x)$$

Its negation is

$$\sim (\forall x \in \mathbf{R})\ (\exists y \in \mathbf{R})\ \ (y^2 = x)$$

$$(\exists x \in \mathbf{R}) \sim (\exists y \in \mathbf{R})\ \ (y^2 = x)$$

$$(\exists x \in \mathbf{R})\ (\forall y \in \mathbf{R}) \sim (y^2 = x)$$

$$(\exists x \in \mathbf{R})\ (\forall y \in \mathbf{R})\ \ (\mathrm{y}^2 \neq \mathrm{x})$$

"There is a real number that does not have a square root."

Next Problem

P4. The square of any irrational number is irrational. [Note the negation lurking in the *ir* part of *ir*rational!]

Note that the word "any" indicates a universal quantifier.

$(\forall x \in \mathbf{R})($ x is not rational $\rightarrow x^2$ is not rational$)$

Its negation is

$\sim (\forall x \in \mathbf{R})(x$ is not rational $\rightarrow x^2$ is not rational$)$

which is

$(\exists x \in \mathbf{R}) \sim (x$ is not rational $\rightarrow x^2$ is not rational$)$

$(\exists x \in \mathbf{R}) (x$ is not rational $\wedge \sim (x^2$ is not rational$)$

$(\exists x \in \mathbf{R}) (x$ is not rational $\wedge\ x^2$ is rational$)$

"There is an irrational number whose square is rational."

(This happens to be true – take $x = \sqrt{2}$, for example.)

Next Problem

P5. A quadratic equation $ax^2 + bx + c = 0$ always has at least one solution.
There are a couple of ways of interpreting this. One is, "For every choice of three real numbers *a*, *b*, and *c*, the equation has a solution." The other is to recall that a quadratic polynomial is a polynomial in which the highest power of the variable that actually occurs in the polynomial is 2, i.e. the coefficient a of x^2 is not zero. Since we're doing this for practice, I'll go with the second interpretation, since it's a bit more complex.

$$(\forall a \in \mathbf{R})(\forall b \in \mathbf{R})(\forall c \in \mathbf{R})\ \ (a \neq 0 \rightarrow (\exists x \in \mathbf{R})(\ ax^2 + bx + c = 0))$$

Its negation is

$$\sim(\forall a \in \mathbf{R})(\forall b \in \mathbf{R})(\forall c \in \mathbf{R})\ (a \neq 0 \rightarrow (\exists x \in \mathbf{R})(\ ax^2 + bx + c = 0))$$

Now apply the rule for negating a universal quantifier three times, and we get

$$(\exists a \in \mathbf{R})(\ \exists b \in \mathbf{R})(\ \exists c \in \mathbf{R})\ \ \sim(a \neq 0 \rightarrow (\exists x \in \mathbf{R})(\ ax^2 + bx + c = 0))$$

Apply the rule for negating a conditional:

$$(\exists a \in \mathbf{R})(\ \exists b \in \mathbf{R})(\ \exists c \in \mathbf{R})\ \ (a \neq 0 \wedge \sim (\exists x \in \mathbf{R})(\ ax^2 + bx + c = 0))$$

$$(\exists a \in \mathbf{R})(\ \exists b \in \mathbf{R})(\ \exists c \in \mathbf{R})\ \ (a \neq 0 \wedge\ \ (\forall x \in \mathbf{R}) \sim (\ ax^2 + bx + c = 0))$$

from which we finally get

$$(\exists a \in \mathbf{R})(\ \exists b \in \mathbf{R})(\ \exists c \in \mathbf{R})\ \ (a \neq 0 \wedge\ \ (\forall x \in \mathbf{R})\ \ (\ ax^2 + bx + c \neq 0))$$

"There is a quadratic equation that all real numbers fail to satisfy."

Next Problem

P6. There is a number z such that if you add z to any number y, you get y.

Since this can be said in regard to any number system, I'll omit the $\in \mathbf{R}$.

$(\exists z)(\forall y)(z + y = y)$

Its negation is

$\sim (\exists z)(\forall y)(z + y = y)$

which is

$(\forall z)(\exists y) \sim (z + y = y)$

$(\forall z)(\exists y) \; (z + y \neq y)$

"For every number z there is a number y such that $z + y$ is different from y."

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