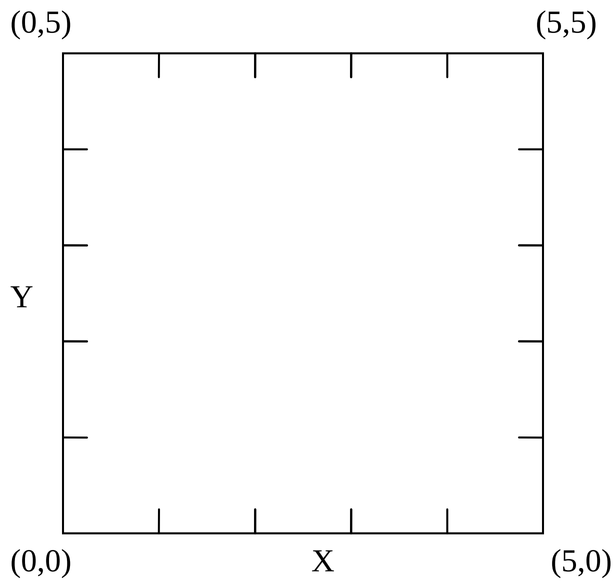
A Geometric Interpretation of Statements with Two Quantifiers

Statements containing two quantifiers are often more difficult to understand than statements containing a single quantifier – especially if the two quantifiers are of different types, that is, one existential and one universal. Statements of the forms “For all x there exists a y such that …” and “There exists a y such that for all x, …” have very different meanings. The purpose of this note is to illustrate the differences in a geometric setting.

For our Universe of Discourse (U), we will take the 5×5 square in the X-Y plane with corners at (0,0), (5,0), (5,5), and (0,5). It is shown below:



A propositional function defined on U defines a truth set, which is a subset of U. We will call it a truth region, because in our examples, the truth set will be a regain in U having a familiar shape.

There are two trivial examples. If we define Boolean functions $\mathcal{T}$ and $\mathcal{F}$ by the rules

$\mathcal{T}(p)$ = true for all points $p \in$ U

$\mathcal{F}(p)$ = false for all points $p \in$ U,

Then the truth region of $\mathcal{T}$ is the entire square and the truth region of $\mathcal{F}$ is the empty set.

We will examine four examples. For each case we will look at the statements

$(\forall\, x)\ (\exists\, y)\ P(x, y)$

$(\exists\, x)\ (\forall\, y)\ P(x, y)$

$(\forall\, y)\ (\exists\, x)\ P(x, y)$

$(\exists\ \mathrm{y})\ (\forall\ \mathrm{x})\ P(x, y)$

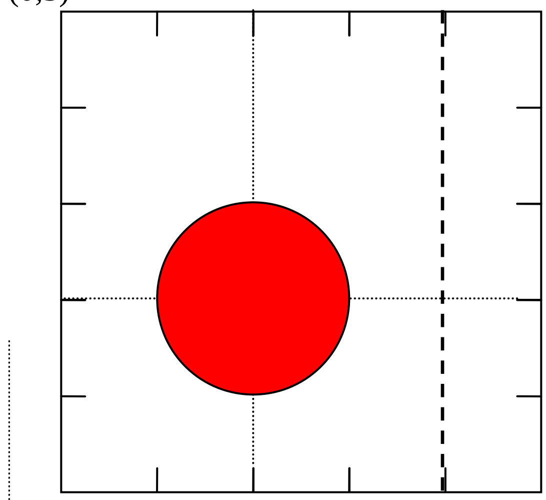
for four different propositional functions P.

Example 1. A small circular region

Suppose we define a propositional function F on U by the rule

F(x, y) is the statement "$(x - 2)^2 + (y - 2)^2 \leq 1$"

The meaning of this statement is, "The distance of the point (x, y) from the point (2, 2) is less than or equal to 1. The truth region of P is then the disk* centered at (2, 2) and having radius 1. It is shown below.



First let's consider the statement $(\forall x)\ (\exists y)$ F(x, y). It means, "For every value of x (i.e every horizontal position), there is at least one point (x, y) that is in the disk. This is not true, as we can see by the counterexample $x = 4$. $x = 4$ defines a vertical line, shown in the above picture by the dashed line. No point on this line is in the truth region.

We can see that the geometric interpretation of $(\forall x)\ (\exists y)$ F(x, y) is, "Every vertical line intersects the truth region."

Now lets look at $(\exists x)\ (\forall y)$ F(x, y). This means that there is at least one value of x such that every point (x, y) is in the truth region. The geometric interpretation of $(\exists x)\ (\forall y)$ F(x, y) is, "There is at least one vertical line that is entirely contained within the truth region."

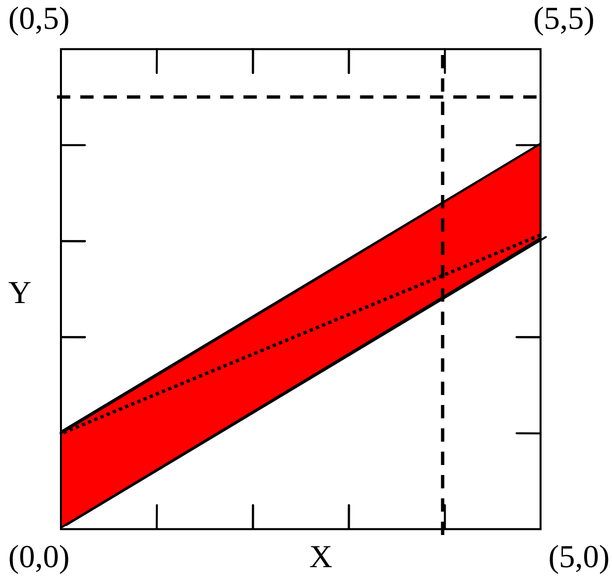
If we interchange the rolls of x and y, we get similar interpretations:

- The statement $(\forall y)\ (\exists x)$ F(x, y) means that every horizontal line intersects the truth region. In our example, this is false.
- The statement $(\exists y)\ (\forall x)$ F(x, y) means that there is at least one horizontal line that is entirely contained within the truth region. In our example, this is false.

(*) A disk is solid; a circle is just a closed curved line. The boundary of a disk is a circle.

Example 2. A diagonal stripe.

Here we let G(x, y) be the statement "$(3/5)x \leq y \leq (3/5)x + 1$". Its truth region is shown below:



The bottom of the region is the line with equation $y = (3/5)x$. The top of the region is the line with equation $y = (3/5)x + 1$.

In this example, the statement $(\forall\, x)\, (\exists\, y)$ G(x, y) is true. Every vertical line intersects the truth region. The vertical line at $x = 4$, shown dashed, is typical.

Another way to describe this condition is: The truth region contains the graph of at least one function y = $f(x)$. In our example, the truth region contains the graph of the function $f_1(x) = (3/5)x$. This graph is the lower boundary of our region. Our region also contains the graph of the function $f_2(x) = (3/5)x + 1$, that being the upper boundry. Another such function is $f_3(x) = 1 + (2/5)x$, which cuts across. The graph of f_3 is shown as a dotted line.

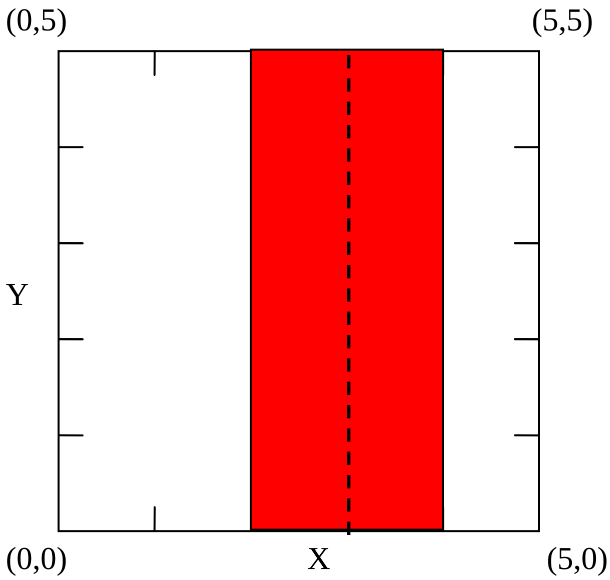
But again, the statement $(\exists\, x)\, (\forall\, y)$ G(x, y) is false. No mater what value you choose for x, there are values for y that give you a point outside the truth region. Our region, like that of Example 1, does not contain any vertical line.

Now let's interchange the rolls of x and y.

- The statement $(\forall\, y)\, (\exists\, x)$ G(x, y) means that every horizontal line intersects the truth region. In our example, this is not true, as shown by the horizontal line at $y = 4.5$ (the dashed horizontal line).
- The statement $(\exists\, y)\, (\forall\, x)$ G(x, y) means that there is at least one horizontal line that is entirely contained within the truth region. This is false too

Example 3. A vertical band.

Here we define H(*x*, y) as the statement "$2 \le x \le 4$". Its truth region is shown below.



Here, (∀ *x*) (∃ *y*) H(*x*, *y*) is false. There are vertical lines that do not intersect the truth region. One such line is the line at $x = 1$. $x = 1$ is a counterexample to the statement (∀ *x*) (∃ *y*) H(*x*, *y*).

(∃ *x*) (∀ *y*) H(*x*, *y*) is true. X =3 provides an example. The entire vertical line at $x = 3$ (dashed) is contained within the truth region.

Now let's we interchange the rolls of *x* and *y*.

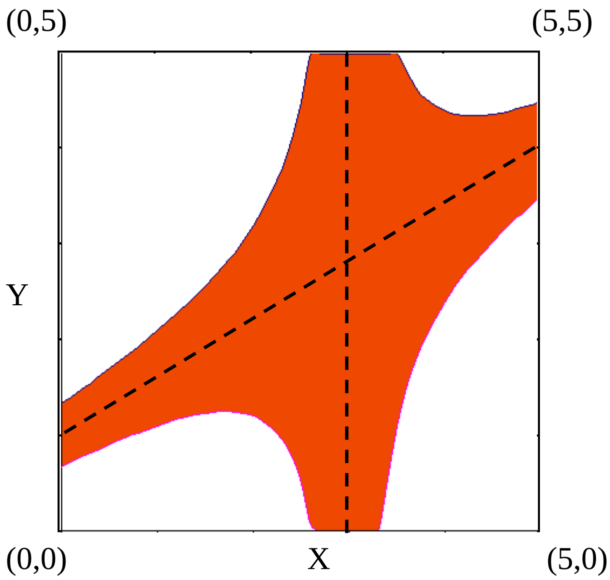
- The statement (∀ *y*) (∃ *x*) H(*x*, *y*) means that every horizontal line intersects the truth region. In our example, this is true
- The statement (∃ *y*) (∀ *x*) H(*x*, *y*) means that there is at least one horizontal line that is entirely contained within the truth region. This is false.

Example 4. A strange-looking region

Let us define K(x, y) to be the statement

$$| (y - (3/5)x - 1)(x - 3) | \leq 1$$

Its truth region is shown below:



Notice the absolute value symbols surrounding the left side of the inequality. The dashed lines are defined by the equations $(y - (3/5)x - 1) = 0$ and $(x - 3) = 0$. The expression on the left side of our inequality is zero along these lines, 1 on the boundary of the region, and greater than 1 outside it.

So let's consider these four statements:

- $(\forall\, x)\ (\exists\, y)$ K(x, y) is true. The region contains the graph of $y = (3/5)x + 1$. Of course, the region also contains the graphs of many other functions, such as $y = (3/5)x + 1.01$.
- $(\exists\, x)\ (\forall\, y)$ K(x, y) is true. $x = 3$ is one such x, as are many others including 2.9 and 3.1.
- $(\forall\, y)\ (\exists\, x)$ K(x, y) is true, just choose $x = 3$ or 2.9 or 3.1 or anything else close to 3, independent of the value of y.
- $(\exists\, y)\ (\forall\, x)$ K(x, y) is false. A value of y specifies a horizontal line, and there is no horizontal line that is entirely contained in our region.