Loaded Quantifiers

In many textbook discussions of quantifiers in logic, it is tacitly assumed that there is a single set X, such as **N** or **Z** or **R**, that is under discussion. But in more realistic situation, there may be several sets involved, including subset of other sets. Many writers use a simple and intuitive notation to deal with these situations. For a starting example, let's consider what it means for a function f: **R** → **R** to be continuous at a point a in **R**. It means:

> For every positive real number ε, there exists a positive real number δ, such that for all real numbers x, if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

So here we are dealing not only with the set **R** of all real numbers but also with the set of all positive real numbers. Symbolically, this would be written

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall x)\ (\ |x - a| < \delta \rightarrow |f(x) - f(a)| < \varepsilon)$$

Notations like $(\forall \varepsilon > 0)$ and $(\exists \delta > 0)$ are what I call *loaded quantifiers*. The purpose of this note is to explain precisely what notations of this type mean. Generally, if you are just reading the statement, the notation is intuitive enough that you don't have to think further. However, if you are now going to disprove a statement of this type about some function, then you first have to form the negation of the statement, and then the precise meaning becomes important.

1. Loaded Universal Quantifiers

The statement "For every x that is greater than zero, …" actually means, "For every x, if x is greater than zero, then….". Let's write this symbolically:

$$(\forall x > 0)\ P(x) \text{ means } (\forall x)\ (\ x > 0 \rightarrow P(x))$$

Note the stealth implication lurking in $(\forall x > 0)\ P(x)$! If we want to write a negation of this statement, we have to take into account the rule for negation of an implication: the negation of $p \rightarrow q$ is $p \wedge \sim q$. So using this fact in combination with the rule for the negation of a universal quantifier, we see that the negation of

$$(\forall x > 0)\ P(x)$$

is

$$(\exists x)\ (x > 0 \wedge \sim P(x))$$

which can be read, "There is a positive number x such that $P(x)$ is false."

Symbolically,

$$(\exists x > 0)\ (\sim P(x))$$

2. Loaded Existential Quantifiers

The statement "There exists an *x* that is greater than zero such that …" actually means, "There exists an *x* such that *x* is greater than zero and …". Lets write this symbolically:

$(\exists x > 0)\ P(x)$ means $(\exists x)\ (x > 0 \wedge P(x))$

This time we have a stealth "and" lurking in $(\exists x > 0)\ P(x)$. If we want to write a negation of this statement, we have to take into account the rule for negation of an "and": the negation of $p \wedge q$ is $\sim p \vee \sim q$. So using this fact in combination with the rule for the negation of an existential quantifier, we see that the negation of

$(\exists x > 0)\ P(x)$

is

$(\forall x)\ (x <= 0 \vee \sim P(x))$

The x <= 0 is essentially saying that values of x that are less than or equal to zero are irrelevant, so the expression is conventionally written

$(\forall x > 0)\ (\sim P(x))$

3. The Bottom Line on Loaded Quantifiers

As far as negations are concerned, loaded quantifiers can be handled just like ordinary quantifiers.

Caution: When changing a "for all" to a "there exists", or vice versa, for a loaded quantifier, *do not change the load.* Thus,

The negation of $(\forall x > 0)\ P(x)$ is $(\exists x > 0)\ (\sim P(x))$

The negation of $(\exists x > 0)\ P(x)$ is $(\forall x > 0)\ (\sim P(x))$

Changing the load would be like changing the domain of a function!