A Very Different Application of Mathematical Induction: The Lattice-Point Polygon Theorem

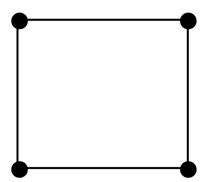
Most of the mathematics you have learned is heavily focused on procedures; very little is taught about strategy. But mathematical induction is not a procedure. It is a strategy, with very broad applicability. Most of the examples you have seen involve proving the correctness of a formula for the sum of a series. But this is a very biased sample of the applicability of mathematical induction.

The Lattice-Point Polygon Theorem (LPPT) is a nice geometric curiosity. Is says: if P is a polygon drawn in the plane with Cartesian coordinates, and if all of the vertices of the polygon are lattice points (points with integer coordinates), then the area of the polygon is

(Number of interior lattice points) + (1/2)(Number of boundary lattice points) – 1

Let's look at two simple examples. Figure 1 shows a 1x1 square. There are no lattice points inside it (zero interior lattice points), and there are four boundary lattice points (the four corners).

Figure 1



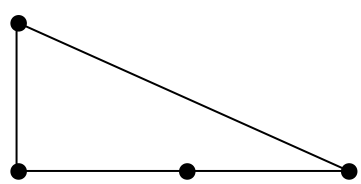
Interior:	0
Boundary:	4
Area:	1

For this square, LPPT says that the area of the square is

$$0 + (1/2)(4) - 1 = 1$$

which is correct. Now look at Figure 2, which is a 1 x 2 triangle. Again there are zero interior lattice points. And there are four boundary lattice points.

Figure 2



Interior:	0
Boundary:	4
Area:	1

So the LPPT says that the area is

$$0 + (1/2)(4) - 1 = 1$$

which is correct.

With some effort (which you should not bother with, as it would be a distraction), it is possible to show that any right triangle with lattice points as vertices satisfies LPPT. And we need another observation: any polygon with lattice point vertices can be constructed from right triangles. (It is sometimes necessary to remove triangles as well as add them.) But again, the details are a distraction, which we will ignore.

Now we are ready to bring mathematical induction to bear on the LPPT.

Let n be a positive integer, and suppose we know that <u>all</u> polygons of area n <u>or less</u> satisfy LPPT. Let P be a polygon with area n+1. We will show that P satisfies LPPT.

Look at Figure 3. It shows a polygon (our P) that has been divided into two smaller polygons by drawing a line between two vertices X and Y. Let

A be the number of vertices in the upper side of P

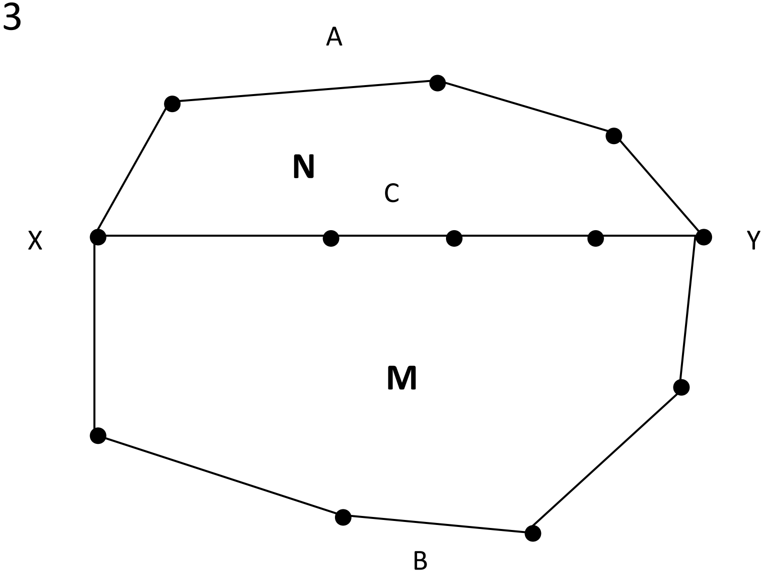
B be the number of vertices in the lower side of P

C be the number of vertices on the dividing line

N be the number of interior lattice points in the upper part of P

M be the number of interior lattice points in the lower part of P

Figure 3



First we have to notice that the number of boundary lattice points of the top section is A + C + 2, the 2 accounting for the points X and Y. Similarly, the number of boundary lattice points of the lower section is B + C + 2. And the number of boundary lattice points of P is A + B + 2. We can also note that the number of interior lattice points of P is equal to N + M + C.

Now both the upper and lower parts have area less than n + 1, that is, each has area less than or equal to n. By the *inductive hypothesis*, we can apply LPPT to each section:

Area of upper part of P = N +(1/2)(A + C + 2) – 1 [a]

Area of lower part of P = M +(1/2)(B + C + 2) – 1 [b]

We have to use this information to show that LPPT applies to P. What does LPPT say about P? It says

$$\text{Area of P} = (N + M + C) + (1/2)(A + B + 2) - 1 \qquad [c]$$

and this statement is our *inductive target*. But [c] follows from [a] and [b] by just adding them! We are done.

Note the interesting variant of the mathematical induction strategy. Instead if proving

$$P(n) \rightarrow P(n+1)$$

we proved

$$P(1) \ \&\& \ P(2) \ \&\& \ \ldots \&\& \ P(n) \rightarrow P(n+1)$$

This strategy is called *Strong Induction*, and it is the strategy very often needed for applications of mathematical induction in computer science.