

Working with Complex Logical Expressions and their Negations

Introduction

The C++ Programming Language, as do all programming languages, includes a variety of comparison operators ($<$, $<=$, $>$, $>=$, $==$, and $!=$) and the logical operators $\&\&$ (AND), $||$ (OR), and $!$ (NOT). Comparison operators operate on two numeric, character, or Boolean values and produce a Boolean result. The logical operators operate on one or two Boolean values and produce a Boolean result.

Comparison and logical operations can be combined to represent complex logical conditions that may be needed for decision-making in computer programs. For example,

$$!(x < y \ \&\& \ (z >= w \ || \ !(z <= t)))$$

includes three comparison operators, one each $\&\&$ and $||$, and two instances of $!$. Such expressions can be difficult to write correctly, difficult to understand, and difficult to debug.

Fortunately, there are some methods that can be used to simplify complex logical expressions and make them more understandable. These methods are called *negation rules*. There are two sets of negation rules: rules for negation of comparison operators, and rules for negation of logical operators. The first set comes from mathematics, the second from logic.

In the following paragraphs, we will explain each of the laws, and give examples showing how they can be used to simplify complex expressions. At the end, there are some recommended exercises, with solutions.

Negation Rules for Comparison Operators

You know from elementary mathematics that if a number is, say, not greater than 12, then it must either be equal to 12 or be less than 12. More generally, if you have two numbers x and y , and if x is not greater than y , then x must be less than or equal to y . This leads us to our first negation rule, which we will write in C++ notation. Suppose E and F are numerical expressions. Then

$!(E > F)$ is equivalent to $E <= F$

Similar reasoning leads to negation rules for each of the other comparison operators. Here they are:

$!(E > F)$	is equivalent to	$E <= F$
$!(E >= F)$	is equivalent to	$E < F$
$!(E < F)$	is equivalent to	$E >= F$
$!(E <= F)$	is equivalent to	$E > F$
$!(E == F)$	is equivalent to	$E != F$
$!(E != F)$	is equivalent to	$E == F$

Example 1. Simplify $!(x \leq y + z)$.

By the fourth rule, this is equivalent to $(x > y + z)$.

Example 2. Simplify $!(x + y * z \neq u * v + w)$

By the sixth rule, this is equivalent to $(x + y * z == u * v + w)$

Negation Rules for Logical Operators

Our first rule is trivial: two negations cancel each other out. More precisely, if **E** is a logical expression, then $!!\mathbf{E}$ is equivalent to **E**.

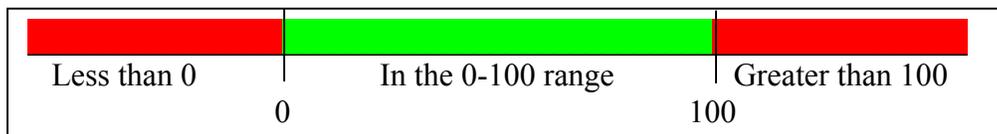
The negation rules for **&&** (AND) and **||** (OR) are much more interesting. They are called DeMorgan's Laws, after the English logician Augustus DeMorgan (1806-1871), who first stated them in formal terms. Here they are:

$!(\mathbf{E} \ \&\& \ \mathbf{F})$	is equivalent to	$(!\mathbf{E}) \ \ (!\mathbf{F})$
$!(\mathbf{E} \ \ \mathbf{F})$	is equivalent to	$(!\mathbf{E}) \ \&\& \ (!\mathbf{F})$

We will illustrate the first with a mathematical example. Consider the condition "g is a number in the range 0 to 100." This is equivalent to "g is greater than or equal to zero, and also g is less than or equal to 100." In C++ notation, we would write

$$(g \geq 0) \ \&\& \ (g \leq 100)$$

Now what does it mean for g to be *outside* the range 0 to 100? This can happen in either of two ways: g can be less than zero, or greater than 100. The situation is shown in the following figure. The two ways that a number can be outside the 0-100 range are shown in red.



Let's write the negation of the C++ expression above. It is

$$!((g \geq 0) \ \&\& \ (g \leq 100))$$

Notice that we enclosed the entire original expression in parentheses before sticking the **!** in front of it! This is very important.

Now we apply the first of the two DeMorgan laws. The law says that our negated expression is equivalent to

$$!(g \geq 0) \ || \ !(g \leq 100)$$

where the expressions **E** and **F** are $(g \geq 0)$ and $(g \leq 100)$ respectively. We can further simplify this by applying the negation laws for \geq and \leq . The result is

$$((g < 0) \ || \ (g > 100))$$

which exactly describes the red portion of the figure.

Example 3. Simplify $!(x \leq 5 \ || \ x \geq 10)$.

By DeMorgan's second law, this is equivalent to

$!(x \leq 5) \ \&\& \ !(x \geq 10)$

Using the negation rules for \leq and \geq , this is equivalent to

$(x > 5) \ \&\& \ (x < 10)$

One More Example

We will use our laws to simplify the (admittedly contrived) example at the start of this note. We will proceed very systematically, applying one law at a time, after inserting parentheses where needed.

Example 4. Simplify $!(x < y \ \&\& \ (z \geq w \ || \ !(z \leq t)))$. Here are the steps:

$!(x < y \ \&\& \ (z \geq w \ \ !(z \leq t)))$	Original
$!((x < y) \ \&\& \ ((z \geq w) \ \ !(z \leq t)))$	Add some parentheses
$!(x < y) \ \ (!(z \geq w) \ \ !(z \leq t))$	DeMorgan's Law for $ $
$!(x < y) \ \ (!(z \geq w) \ \&\& \ !(z \leq t))$	DeMorgan's Law for $\&\&$
$!(x < y) \ \ (!(z \geq w) \ \&\& \ (z \leq t))$	Negation of $<$
$(x \geq y) \ \ ((z < w) \ \&\& \ (z \leq t))$	Negation of \geq

It is extremely unlikely that you will ever encounter anything half this complex. Example 3 is far more typical. But the above example, aside from providing practice with the laws, does illustrate how we deal with complexity in computing: one bite at a time.

Recommended Exercises

(Solutions are on the following page.)

Exercise 1. Simplify $!(x == 113 \ \&\& \ y != 2)$

Exercise 2. Simplify $!(height < 100 \ || \ width > 275)$

Exercise 3. Simplify

$!(temperature \geq 100 \ \&\& \ pressure \geq 20 \ \&\& \ volume < 250)$

Exercise 4. Simplify

$!(cost \leq price \ || \ price \geq tax \ || \ tax \leq income * 0.5)$

Solutions to Exercises

Exercise 1. Simplify $!(x == 113 \ \&\& \ y \neq 2)$
 $(x \neq 113 \ || \ y == 2)$

Exercise 2. Simplify $!(\text{height} < 100 \ || \ \text{width} > 275)$
 $(\text{height} \geq 100 \ \&\& \ \text{width} \leq 275)$

Exercise 3. Simplify
 $(\text{temperature} < 100 \ || \ \text{pressure} < 20 \ || \ \text{volume} \geq 250)$

Exercise 4. Simplify
 $(\text{cost} > \text{price} \ \&\& \ \text{price} < \text{tax} \ \&\& \ \text{tax} > \text{income} * 0.5)$